

Wave Hierarchies in Viscoelasticity

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Abstract- An evolution operator \mathcal{L}_n with n arbitrary, typical of several models, is analyzed. When $n = 1$ the operator characterizes the Standard Linear Solid of viscoelasticity, whose properties are already established in previous papers. The fundamental solution \mathcal{E}_n of \mathcal{L}_n is explicitly obtained and it's estimated in terms of the fundamental solution \mathcal{E}_1 of \mathcal{L}_1 . So, whatever n may be, asymptotic properties and maximum theorems are achieved. These results are applied to *the Rouse model* and *reptation model*, which describe different aspects of polymer chains.

Keywords- Viscoelasticity, Wave hierarchies, Partial differential equation, Maximum principles.

1. INTRODUCTION

Let \mathcal{L}_n be the $(2 + n)$ order operator:

$$\mathcal{L}_n = \partial_t^{(n)}(\partial_{tt} - c_n^2 \partial_{xx}) + a_{n-1} \partial_t^{(n-1)}(\partial_{tt} - c_{n-1}^2 \partial_{xx}) + \dots \dots a_0 (\partial_{tt} - c_0^2 \partial_{xx}) \quad (1.1)$$

where a_k ($k = 0..n - 1$) are positive constants.

According to the value of n , (1.1) describes several physical phenomena. As example, when $n = 1$, \mathcal{L}_n can be found in dynamic of relaxing gases, in magnetohydrodynamics, in hereditary electromagnetism (see [1] and references therein) and in isotropic viscoelasticity where, (1.1) models the evolution of the Standard Linear Solid (S.L.S.) (see, f.i. [2, 3]).

In all these models c_k 's represent the characterized speeds depending on the materials properties of the medium and in many physical problems it results $c_0^2 \leq c_1^2 \leq c_{n-1}^2 \leq c_n^2$ as it's typical of *wave hierarchies*. [4].

When $n = 1$, the operator (1.1) is strictly-hyperbolic and it has been widely analyzed in [1]. It's fundamental solution \mathcal{E}_1 has been explicitly determined and singular perturbation problems, together with asymptotic properties, have been estimated.

Aim of the paper is to draw generalizations of the wide analysis related to S.L.S. to the case of (1.1) with n arbitrary. For this, a *conditioned* equivalence between (1.1) and an integro - differential operator \mathcal{M} related to an appropriate memory function $g_n(t)$, is considered. Owing to the hypotheses of fading memory, every function $g_n(t)$ can be approximated by Dirichlet polynomials [5, 6] with appropriate restrictions on the

coefficients of this expansion. These limitations need that the differential operator \mathcal{L}_n is typical of *wave hierarchies* [7, 8].

By this equivalence, whatever n may be, the fundamental solution \mathcal{E}_n of (1.1) is explicitly achieved and it is estimated in terms of \mathcal{E}_1 . So the maximum properties and asymptotic estimates established for the Standard Linear Solid can be applied to operator \mathcal{L}_n defined in (1.1). Moreover, boundary layer problems, typical of dissipative media, can be rigorously estimated. [9, 10].

These results are applied to *the Rouse model* and *the reptation model* which describe different aspects of polymer chains and have met with reasonable success.[11]-[13].

2. DIFFERENTIAL CONSTITUTIVE EQUATION

Let \mathcal{B} a linear, isotropic, homogeneous system and let $\underline{u}(\underline{x}, t)$ the displacement field from an undeformed reference configuration \mathcal{B}_0 . If $\underline{u} = u(x, t) \underline{i}$ and ρ_0 denotes the mass density in \mathcal{B}_0 , the equations of one-dimensional motions of \mathcal{B} are

$$\rho_0 u_{tt} = \sigma + f, \quad \varepsilon = u, \quad (2.1)$$

where $\underline{f} = f \underline{i}$ is the known body force, while σ and ε are the only non vanishing components of the stress and the strain tensors.

When the viscoelastic behavior of \mathcal{B} is of *rate-type*, the well known stress-strain constitutive relation is

$$\sum_{k=0}^n a_k \partial_t^k \sigma = \sum_{k=0}^n \alpha_k \partial_t^k \varepsilon \quad (2.2)$$

with a_k, α_k constant ($a_n, \alpha_n \neq 0$).

Then, by (2.1), (2.2), the displacement field $\underline{u}(\underline{x}, t)$ is solution of the higher order equation like:

$$\mathcal{L}_n v \equiv \sum_{k=0}^n a_k \partial_t^k (v_{tt} - c_k v_{xx}) = F \quad (2.3)$$

where:

$$c_k = \alpha_k / \rho_0 a_k, \quad F = (1/\rho_0) \sum_{k=0}^n a_k \partial_t^k f. \quad (2.4)$$

The constitutive relation (2.2) includes various classical mechanical models, as Maxwell and Kelvin - Voigt models [14]. Moreover, when $n = 1$ and

$$0 < c_0 < c_1, \quad \eta = a_1/a_0 > 0 \quad (2.5)$$

one has the case of the *Standard-Linear Solid* (S.L.S.) which is modelled by the strictly-hyperbolic third order equation:

$$\mathcal{L}_1 v \equiv \eta \partial_t (v_{tt} - c_1 v_{xx}) + v_{tt} - c_0 v_{xx} = (1/a_0) F. \quad (2.6)$$

The fundamental solution \mathcal{E}_1 of this operator has been obtained in [1], also when $\underline{x} \in R^2$ or $\underline{x} \in R^3$. Further, numerous basic properties of \mathcal{E}_1 have been rigorously estimated and the wave behavior of S.L.S. is now acquired.

When $n > 1$ and all the c_k 's are positive, then waves of different orders appear and their roles must be clarified in order to see how each set is modified by the presence of the other. Obviously, wave or dispersive behavior depend on the requirements of the coefficients a_k and c_k due to physical properties of the system \mathcal{B} .

For this, we will analyze the restrictions imposed on the constants a_k and c_k by usual hypotheses of fading memory for \mathcal{B} .

3. INTEGRAL CONSTITUTIVE EQUATION

When the strain amplitudes are not too large, the behavior of most viscoelastic media is fairly well modelled by linear hereditary equations like:

$$\varepsilon(t) = J(0)[\sigma(t) + \int_{-\infty}^t \dot{J}(t - \tau)\sigma(\tau)d\tau], \quad (3.1)$$

where $J(t)$ denotes the creep-compliance and the integral term needs the knowledge of the past history of the stress.

Usually, according to *fading memory* hypotheses, $\dot{J}(t)$ is a positive fast decreasing function. For instance, several real materials as polymers, rubbers, bitumines, have satisfactory representations by means of chains of S.L.S. elements in series or parallel [3, 11]. In the series case, the creep function is

$$J_n(t) = J_n(0)[1 + \sum_{k=1}^n \frac{B_k}{\beta_k}(1 - e^{-\beta_k t})], \quad (3.2)$$

where n is the number of elements in the chain, $\tau_k = \beta_k^{-1}$ are the characteristic times and $J_n(0)$ denotes the elastic compliances.

Then, if one puts:

$$c^2 = [\rho_0 J_n(0)]^{-1}, \quad F_* = c^2 [J_n(0)f + \int_{-\infty}^0 \dot{J}_n(t - \tau)\sigma_x(\tau)d\tau], \quad (3.3)$$

by (2.1), (3.1), (3.2) one deduces:

$$\mathcal{M}u \equiv c^2 u_{xx} - u_{tt} - \int_0^t g(t - \tau) u_{\tau\tau} d\tau = -F_*(x, t), \quad (3.4)$$

with

$$g = g_n(t) = \sum_{k=1}^n B_k e^{-\beta_k t} = \dot{J}_n(t)/J_n(0). \quad (3.5)$$

In this memory function, n is quite arbitrary and constants B_k and frequencies β_k are such that:

$$0 < \beta_1 < \beta_2 \dots < \beta_n; \quad B_k > 0 \quad \forall k = 1, 2, \dots, n. \quad (3.6)$$

These hypotheses assure that:

$$g(t) > 0, \quad \dot{g} < 0, \quad \ddot{g} > 0, \quad \forall t \geq 0 \quad (3.7)$$

according to the convexity assumption considered by Dafermos [15].

We observe that the representation (3.5) of the memory function g is not restrictive because well-known Muntz and Schawart'z theorems [5, 6] imply that whatever $C^0(R^+)$ function can be uniformly represented by means of Dirichlet polynomials. Moreover, as n is arbitrary, the constants B_k, β_k can be determined in order to fit the experimental curves for $g(t)$ to any prefixed degree of approximation [11].

By (3.2), (3.6) one has

$$J_n(\infty) = J_n(0) \left[1 + \sum_{k=1}^n \frac{B_k}{\beta_k} \right] > J_n(0). \quad (3.8)$$

4. FADING MEMORY AND WAVE HIERARCHIES

The initial data related to (2.3) and (3.4) let be null and let

$$P(s) = \sum_{k=0}^n \mu_k s^k, \quad Q(s) = \sum_{k=0}^n \lambda_k s^k \quad (4.1)$$

with

$$\mu_k = a_k/a_n, \quad \lambda_k = a_k c_k/a_n c_n \quad (k = 0, \dots, n) \quad (4.2)$$

so that $\mu_n = \lambda_n = 1$.

Further, let

$$G(s) = \sum_{k=1}^n \frac{B_k}{s + \beta_k} \quad (4.3)$$

the Laplace transform of the memory function (3.5).

Then, if one applies the Laplace transformation to (2.3) and (3.4), it results:

$$\hat{v}_{xx} - \frac{s^2}{c_n} \frac{P(s)}{Q(s)} \hat{v} = -\frac{1}{a_n c_n} \frac{\hat{F}}{Q(s)} \quad (4.4)$$

$$\hat{u}_{xx} - \frac{s^2}{c^2} [1 + G(s)] \hat{u} = -\frac{\hat{F}_*}{c^2} \quad (4.5)$$

where $(\hat{})$ denote the L -transform of () .

By comparing (4.4), (4.5) one deduces

$$\frac{P(s)}{Q(s)} = \frac{c_n}{c^2} [1 + G(s)] \quad (4.6)$$

and the polinomial identity implies $c_n = c^2$ and

$$\begin{cases} \lambda_0 = \beta_1 \beta_2 \dots \beta_n \\ \dots\dots\dots \\ \lambda_{n-2} = \beta_1 \beta_2 + \beta_1 \beta_3 + \dots \beta_{n-1} \beta_n \\ \lambda_{n-1} = \beta_1 + \dots + \beta_n \end{cases} \quad (4.7)$$

So, owing to (3.6), all the λ_k 's are positive. Further, as for μ_k , one has:

$$\begin{cases} \mu_0 = \lambda_0 + B_1(\beta_2 \dots \beta_n) + \dots B_n(\beta_1 \dots \beta_{n-1}) \\ \dots\dots\dots \\ \mu_{n-2} = \lambda_{n-2} + B_1(\beta_2 + \dots + \beta_n) + \dots B_n(\beta_1 + \dots + \beta_{n-1}) \\ \mu_{n-1} = \lambda_{n-1} + B_1 + \dots + B_n \end{cases} \quad (4.8)$$

and (3.6), (4.8) imply too: $0 < \lambda_k < \mu_k$ ($k = 0, \dots, n-1$). As consequence:

$$0 < c_k < c_n = c^2 \quad (k = 0, \dots, n-1) \quad (4.9)$$

At last, by (4.7), (4.8), it follows: $\frac{\lambda_0}{\mu_0} < \frac{\lambda_1}{\mu_1} < \frac{\lambda_{n-1}}{\mu_{n-1}}$ and so

$$0 < c_0 < c_1 \dots < c_n. \quad (4.10)$$

So, the following property holds:

Property 4.1. *Hypotheses of fading memory (3.5) (3.6) imply that the differential operator (2.3) is typical of wave hierarchies.* ■

Vice versa, the inverse transformation of (4.7),(4.8) requires carefulness. When the differential equation (2.3) is prefixed, in order to obtain the dual hereditary equation (3.4) with a memory function $g(t)$ satisfying (3.5), (3.6), appropriate restrictions on the constants a_k, c_k must be imposed.

At first, (4.3), (4.6) imply

$$\frac{P(s)}{Q(s)} = B_0 + \sum_{k=1}^n \frac{B_k}{s + \gamma_k} \quad (4.11)$$

where all the roots $s = -\gamma_k$ di $Q(s)$ are real and simple, with $\gamma_k > 0$. Moreover the conditions $B_k > 0$, which are sufficient to verify (3.7), involve further limitations.

Example 4.1 - When $n = 1$, one has: $c^2 = c_1, B_0 = 1$, and

$$\beta_1 = \frac{a_0 c_0}{a_1 c_1} > 0, \quad B_1 = \frac{a_0}{a_1} \left(1 - \frac{c_0}{c_1}\right) > 0 \quad (4.12)$$

which represent the known restrictions typical of S.L.S. ■

Example 4.2 - When $n = 2$, one has: $c^2 = c_2, B_0 = 1$, and β_1, β_2 are real iff:

$$\omega^2 = (a_1 c_1)^2 - 4(a_0 c_0)(a_2 c_2) > 0 \quad (4.13)$$

Then, it results:

$$\beta_1 = \frac{1}{2a_2 c_2} (a_1 c_1 - \omega), \quad \beta_2 = \frac{1}{2a_2 c_2} (a_1 c_1 + \omega) \quad (4.14)$$

so that $0 < \beta_1 < \beta_2$. Further

$$B_i = \frac{(-1)^{i-1}}{\omega} [a_0(c_2 - c_0) - a_1 \beta_i (c_2 - c_1)]. \quad (i = 1, 2) \quad (4.15)$$

Thus, it is $B_1 > 0, B_2 > 0$ iff

$$\beta_1 < \frac{a_0}{a_1} \frac{c_2 - c_0}{c_2 - c_1} < \beta_2. \quad (4.16)$$

Therefore, the fourth-order operator

$$a_2(u_{tt} - c_2 u_{xx})_{tt} + a_1(u_{tt} - c_1 u_{xx})_t + a_0(u_{tt} - c_0 u_{xx}) \quad (4.17)$$

can be analyzed by (3.4), (3.5), (3.6) when the constants a_k, c_k satisfy (4.13) and (4.16). ■

5. ESTIMATES FOR THE HEREDITARY MODEL

Let \mathcal{B}_n the viscoelastic model characterized by the memory function g_n in (3.5); the case $n = 1$ corresponds to the L.S.L. \mathcal{B}_1 .

In [16, 17], the fundamental solution E_n of the operator \mathcal{M} in (3.4) has been explicitly determined, whatever n may be. If η is the step - function and I_0 is the modified Bessel function of first kind, it results:

$$E_n = E_n(\beta_1.. \beta_n, B_1.. B_n) = \frac{1}{2c} \eta(t-r)(A_1 + A_2) \quad (5.1)$$

with

$$A_1 = e^{-g_0 t/2} I_0\left(\frac{g_0}{2} \sqrt{t^2 - r^2}\right) \quad (5.2)$$

$$A_2 = \frac{1}{\pi} \int_0^\pi d\theta \int_r^t e^{-g_0 z} H(z, t-w) du, \quad (5.3)$$

and $g_0 = g(0)$, $r = |x|/c$, $2z = w - \cos\theta (w^2 - r^2)^{1/2}$. Futher, if

$$\phi_k(z, t) = e^{-\beta_k t} \sqrt{B_k \beta_k z/t} I_1(2\sqrt{B_k \beta_k z t}), \quad (5.4)$$

one has:

$$H(z, t) = \sum \phi_k + \sum_{k_1, K_2} \phi_{k_1} * \phi_{k_2} + \dots \quad (5.5)$$

where sums are computed according to the simple combination of the indices $k_1 k_2 .. k_n$ and $*$ denotes the convolution with respect to t .

Moreover the fundamental solution E_n related to \mathcal{B}_n and defined in (5.1)-(5.5), can be rigorously estimated in terms of the fundamental solution E_1 related to an appropriate S.L.S. \mathcal{B}_1^* defined by

$$g_1 = b e^{-\beta_1 t} \quad \text{with} \quad b = \beta_1 \sum_1^n \frac{B_k}{\beta_k}. \quad (5.6)$$

In fact, if Γ is the open forward characteristic cone $\{(t, x) : t > 0, |x| < ct\}$, and $\chi_n = \prod_{k=2}^n (\frac{B_k}{\beta_1})^2$, then the following theorem holds:

Theorem 5.1 - *If the memory function is given by (3.5) (3.6), then the fundamental solution E_n of \mathcal{M} is a never negative $C^\infty(\Gamma)$ function and it satisfies the estimate:*

$$0 < E_n(\beta_1.. \beta_n, B_1.. B_n) < \chi_n E_1(\beta_1, b), \quad (5.7)$$

everywhere in Γ and whatever n may be. ■

Remark 5.1 - The model \mathcal{B}_1^* defined by (5.6) is physically meaningful. In fact the memory function g_1 is related just to the obliuator because $\tau_1 = \beta_1^{-1}$ is the longest characteristic time.

Furthermore \mathcal{B}_n and \mathcal{B}_1^* verify the same hypotheses of fading memory and by (3.5) (5.6), it results:

$$\int_0^\infty g_n(t)dt = \int_0^\infty g_1(t)dt = \sum_1^n \frac{B_k}{\beta_k}. \quad (5.8)$$

Moreover, the integral (5.8) affects the asymptotic analysis of hereditary equation [7, 18]. ■

Remark 5.2. - By known properties of asymptotic behaviour of convolutions, the constitutive relation (3.1) implies:

$$\lim_{t \rightarrow \infty} \varepsilon(x, t) = J_n(0) \left[1 + \int_0^\infty g_n(t)dt \right] \lim_{t \rightarrow \infty} \sigma(x, t) \quad (5.9)$$

provided that $\lim_{t \rightarrow \infty} \sigma$ exists. Then \mathcal{B}_n and the model \mathcal{B}_1^* exhibit the same asymptotic behaviour (5.9)

Further, by (3.5), (3.8) it results:

$$\frac{J_n(\infty)}{J_n(0)} = 1 + \sum_1^n \frac{B_k}{\beta_k} = \frac{J_1(\infty)}{J_1(0)} \quad (5.10)$$

and so, (5.10) implies $J_n(\infty)\sigma(\infty) = \varepsilon(\infty)$. Consequently, when t is large, the behaviour of \mathcal{B}_n is typical of an elastic material with modulus $J_n(\infty)$. ■

6. ESTIMATES RELATED TO WAVE HIERARCHIES

When the operator \mathcal{L}_n is reduced to the hereditary operator \mathcal{M} of (3.4), then estimates of Theorem 5.1 can be applied to wave hierarchies. Obviously, the equivalence is conditioned by inverse transformation of (4.7)- (4.8) together with (3.6) (see n.4).

Let \mathcal{L}_1^* the operator (2.6) related to the S.L.S. \mathcal{B}_1^* characterized by

$$\eta = \frac{1}{\beta_1 + b}, \quad a_0 = \frac{\beta_1 + b}{\beta_1}, \quad c_0 = \frac{c^2 \beta_1}{\beta_1 + b} \quad c_1 = c^2. \quad (6.1)$$

Now, let \mathcal{L}_n the differential operator given by (2.3) whatever n may be, and let \mathcal{P}_n a prefixed boundary value problem related to \mathcal{L}_n . The meaningful aspects of qualitative analysis of the solution of \mathcal{P}_n can be obtained by means of Theorem 5.1 and by the known properties of \mathcal{L}_1^* [1].

So maximum properties, asymptotic behaviour, boundary layer estimates, etc. for the solution of \mathcal{P}_n are deduced by analogous properties related to \mathcal{L}_1^* .

Moreover, owing to the equivalence between \mathcal{L}_n and \mathcal{M} , it is possible to have explicitly the fundamental solution of \mathcal{L}_n for all n . In fact it suffices to apply the explicit formula (5.1)-(5.3).

As an example the case of *polymeric materials* can be considered.

Example 6.1 - Polymeric materials are very flexible and are easily formed into fibres, thin films, additives for oils, etc. So their applications to concrete problems are numerous.[12, 19, 20]. According to theories of linear viscoelasticity, two models, that describe different aspects of polymer chains, have met a reasonable success: *the Rouse model* and *the reptation model* [13].

In both cases the memory function $g(t)$ assumes a form like (3.2)(3.3). In fact in *the reptation model*, the stress relaxation function is:

$$g(t) = k \sum_{h=0}^n \frac{1}{(2h+1)^2} e^{-(2h+1)^2 \frac{t}{\tau_d}}, \quad (6.2)$$

where k is a constant depending on the polymer physics and the value of the "reptation" time τ_d can be fixed according to elasticity experiments [11].

When the viscoelastic behaviour is represented by *the Rouse model*, memory function $g(t)$ is given by :

$$g(t) = k_1 \sum_{h=1}^n e^{2h^2 \frac{t}{\tau_1}}, \quad (6.3)$$

where the relaxation time τ_1 can be calculated by means of experimental results.[13].

So, if one considers the first two steps in *the reptation model*, it results: $B_1 = k$, $B_2 = B_1/9$, $\beta_1 = 1/\tau_d$, $\beta_2 = 9\beta_1$. Consequently the operator (4.17) is characterized by constants:

$$\begin{cases} c_0 = c^2 \frac{81}{81+82k\tau_d} & c_1 = c^2 \frac{9}{9+k\tau_d} & c_2 = c^2 \\ a_0 = 1 + \frac{82}{81} k\tau_d & a_1 = \frac{10\tau_d^2}{9} \left(\frac{1}{\tau_d} + \frac{k}{9} \right) & a_2 = \frac{\tau_d^2}{9}. \end{cases} \quad (6.4)$$

Analogously, in *the Rouse model*, beeing $B_1 = B_2 = k_1$, $\beta_1 = 2/\tau_1$, $\beta_2 = 4\beta_1$, one has:

$$\begin{cases} c_0 = c^2 \frac{8}{8+5k_1\tau_1} & c_1 = c^2 \frac{5}{5+k_1\tau_1} & c_2 = c^2 \\ a_0 = 1 + \frac{5k_1}{8} \tau_1 & a_1 = \frac{\tau_1^2}{16} (2k_1 + \frac{10}{\tau_1}) & a_2 = \frac{\tau_1^2}{16}. \end{cases} \quad (6.5)$$

The *wave hierarchies* defined by (6.4) or (6.5) are governed by the operator \mathcal{L}_1^* of the Standard Linear Solid defined, respectively, by:

$$\begin{cases} c_0 = c^2 \frac{81}{81+82k\tau_d} & c_1 = c^2 & a_0 = 1 + \frac{82}{81} k\tau_d & \eta = \frac{81\tau_d}{81+82k\tau_d}, \\ c_0 = c^2 \frac{8}{8+5k_1\tau_1} & c_1 = c^2 & a_0 = 1 + \frac{5}{8} k_1\tau_1 & \eta = \frac{4\tau_1}{8+5k_1\tau_1}. \end{cases} \quad (6.6)$$

■

These results have been confirmed also in [21] for entangled polymers with chain stretch.

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